Plane R-curves and their steepest descent properties I

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Plane R-curves and their steepest descent properties I

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Let Γ_R be the class of plane, oriented, rectifiable curves γ, such that for almost every $x \in \gamma$, the part of γ preceding x is outside of the open circle of radius R, centered in $x + R\mathbf{t}_x$, where $\mathbf{t}_x$ is the unit tangent vector at x. Geometrical properties of the curves $\gamma \in \Gamma_R$ are proved; it is shown also that the length of a regular curve $\gamma \in \Gamma_R$ is bounded by a constant depending upon R and the diameter of γ only. The curves $\gamma \in \Gamma_R$ turn out to be steepest descent curves for real valued functions with sublevel sets of reach greater than R.

1. Introduction

Let $R > 0$. Let Γ_R be the class of the plane oriented rectifiable curves γ, starting at a point x_0 and such that for every $x \in \gamma$, for which there exists the unit tangent vector $\mathbf{t}_x$, the arc $\gamma_{x_0,x}$ (joining x_0 to x) is not contained in the open circle centered at $x + R\mathbf{t}_x$ and of radius R. In the present work the curves $\gamma \in \Gamma_R$ will be called *R-curves*, for short.

The family Γ_R is a generalization of the family Γ of plane curves studied in [1, 2]. Γ is the class of curves γ for which $\gamma_{x_0,x}$ is contained in the half-plane bounded by the line through x, orthogonal to $\mathbf{t}_x$. The class Γ has also been studied starting from equivalent definitions, in [3–5]. The curves of Γ are steepest descent curves for a suitable family of nested convex sets [6]. Similarly it can be observed (Theorem 6.2) that a function with sublevel sets of reach greater than R has steepest descent curves in Γ_R. Obviously for every $R > 0$, $\Gamma \subset \Gamma_R$.

Many natural questions arise for the curves of Γ_R.

(A) Let x a point of γ, $x \neq x_0$. Let us consider the following number naturally associated to $\gamma_{x_0,x}$:

$$\frac{\text{length}(\gamma_{x_0,x})}{\text{dist }(x_0, x)},$$

called the detour of the curve $\gamma_{x_0,x}$, see e.g. [2, 7]. It has been proved in [1–3] that if $\gamma \in \Gamma$ then, the detour of $\gamma_{x_0,x}$ has an a priori bound. Could a similar result be proved for the curves in Γ_R?

(B) Let $\gamma \in \Gamma_R$ and let's assume that γ is contained in a circle of radius $\tau > 0$; is it possible to bound the length and the detour of γ with a number depending on R, τ only?

(C) Let $\gamma \in \Gamma_R$. Are there functions that have steepest descent curves in Γ_R?

Let us outline the content of our work.

In §2 introductory definitions are given and results on sets of positive reach, needed later, are recalled.

In §3 the definition of R-curves in $\mathbb{R}^n$ is given and several properties are proved.

In §4 properties of plane R-curves in small circles are stated and proved; the main one is a sharp bound of the measure of the tangent angle to the hull of $\gamma_{x_0,x}$ at x, Theorems 4.3 and 4.4. In these theorems, it has been assumed an additional regularity hypothesis: that is γ has a C^1 parametric representation. This assumption will be removed in a paper in

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progress [8]. Theorem 4.4 proves that plane regular R-curves, in a small circle, are ϕ-self approaching curves (with opposite orientation), for a suitable $\phi < \pi$, see [7].

A bound, depending on R only, for the length and the detour of a plane R-curve γ in a small circle is obtained, Theorem 4.5, Theorem 4.7.

In §5 R-curves contained in a circle of fixed arbitrary radius τ are studied. A bound of their length and their detour depending only on R, τ is proved, Theorem 5.2, Theorem 5.3. In Proposition 5.4 an example shows that if a curve of Γ_R is not contained in a circle of fixed radius τ then the detour can be arbitrarily large.

In §6 functions defined in bounded sets of $\mathbb{R}^2$, with the property that their steepest descent curves are R-curves, are considered; by Theorem 5.2 it follows that these steepest descent curves have length bounded by a constant depending on R and their diameter.

A first example of functions with the above property is the class of the regular functions whose sublevel sets satisfy the property of R-exterior ball (Definition 2), Theorem 6.1. The family of regular functions whose sublevel sets are of reach greater than R also satisfies the same property, Theorem 6.2.

A differential property for regular functions, in order to satisfy this property, is that these functions must have as domain a closed bounded connected subset of a circle of radius R and level lines with curvature greater or equal than $-1/R$, Theorem 6.7.

2. Definitions and preliminaries

Let

$$B(z,\rho) = \{x \in \mathbb{R}^n \,:\, |x-z| < \rho\}\,, \quad S^{n-1} = \partial B(0,1) \quad n \geq 2$$

and let $D(z,\rho)$ be the closure of $B(z,\rho)$. The notations B_ρ, D_ρ will also be used for balls of radius ρ, if no ambiguity arises for their center. If a ball is written with a center only, then the radius will be R. The usual scalar product between vectors $u, v \in \mathbb{R}^n$ will be denoted by $\langle u, v\rangle$.

Let $K \subset \mathbb{R}^n$; $Int(K)$ will be the interior of K, ∂K the boundary of K, $cl(K)$ the closure of K, $K^c = \mathbb{R}^n \setminus K$.

For every set $S \subset \mathbb{R}^n$, $co(S)$ is the convex hull of S.

Let K be a non empty closed set. Let $q \in K$; the *tangent cone* of K at q is defined as

$$\mathrm{Tan}_K(q) = \{v \in \mathbb{R}^n : \forall \varepsilon > 0,\, \exists\, x \in K,\ r > 0 \quad \text{with} \quad |x-q| < \varepsilon,\ |r(x-q) - v| < \varepsilon\}.$$

Let us recall that

$$S^{n-1} \cap \mathrm{Tan}_K(q) = \bigcap_{\varepsilon > 0} cl(\{\frac{x-q}{|x-q|},\, q \neq x \in K \cap B(q,\varepsilon)\}).$$

The *normal cone* at q to K is the non empty closed convex cone, given by:

$$\mathrm{Nor}_K(q) = \{u \in \mathbb{R}^n : \langle u, v\rangle \leq 0 \quad \forall v \in \mathrm{Tan}_K(q)\}.$$

When $q \in Int(K)$, then $\mathrm{Tan}_K(q) = \mathbb{R}^n$ and $\mathrm{Nor}_K(q)$ reduces to zero. In two dimensions cones will be called angles with vertex 0.

The *dual cone* K^* of a set K is $K^* = \{y \in \mathbb{R}^n : \langle y, x\rangle \geq 0 \quad \forall x \in K\}$.

For A, B non empty sets of $\mathbb{R}^n$, $x \in \mathbb{R}^n$, let us denote

$$\mathrm{dist}\ (x, A) = \inf_{y \in A}\{|x-y|\}; \quad A^{(\varepsilon)} = \{x \in \mathbb{R}^n : \mathrm{dist}\ (x, A) \leq \varepsilon\}.$$

Let $b \in \mathbb{R}^n \setminus A$; then b has a unique projection point onto A if there exists a unique point $a \in A$ satisfying $|b-a| = \mathrm{dist}\ (b, A)$.

Let A be a closed set. If $a \in A$, then $reach(A, a)$ is the supremum of all numbers ρ for which every $x \in B(a, \rho)$ has a unique projection point onto A. Also, see [9]:

$$reach(A) := \inf\{reach(A, a) : a \in A\}.$$

PROPOSITION 2.1 *[9, Theorem 4.8, (12)] If $a \in A$ and $reach(A, a) > 0$ then*

$$\mathrm{Tan}_A(a) = -(\mathrm{Nor}_A(a))^*.$$

Definition 1 Let us say that a ball B_R *R-supports* A at x, if

$$B_R \cap A = \emptyset, \quad x \in A \cap \partial B_R.$$

If B_R R-supports A at x, then the point x is necessarily a boundary point of A. B_R will be called an *R-support ball* to A at x.

Let us consider a closed set A such that for each point $y \in \partial A$ the normal cone $\mathrm{Nor}_A(y)$ is not reduced to zero.

Definition 2 Let $R > 0$. A closed set A, satisfying the above property, has the *property of the R-exterior ball* if $\forall y \in \partial A$, $\forall n_A(y) \in \mathrm{Nor}_A(y) \cap S^1$, the following fact holds

$$\forall x \in A \Longrightarrow |x - (y + Rn_A(y))| \geq R. \tag{1}$$

The previous definition implies that for every point $y \in \partial A$ there is at least an R-support ball to A at y.

PROPOSITION 2.2 *[9, Theorem 4.8, (7)] If a closed set A has reach greater than R, then it satisfies the property of R-exterior ball.*

Let $x, y \in \mathbb{R}^n, |x - y| < 2R$. Let us define

$$\mathfrak{h}(x, y, R) = \cap\{D_R(z), z \in \mathbb{R}^n : D_R(z) \supset \{x, y\}\}. \tag{2}$$

PROPOSITION 2.3 *[10],[11, Theorem 3.8] A closed set A has reach equal or greater than R if and only if $\forall x, y \in A$ with $|x - y| < 2R$ the set $A \cap \mathfrak{h}(x, y, R)$ is connected.*

Definition 3 Given A a closed set in $\mathbb{R}^n$, let us define $co_R(A)$, the *R-hull* of A, as the closed set containing A, such that
(i) $co_R(A)$ has reach greater or equal than R;
(ii) if a set $B \supseteq A$ and $reach(B) \geq R$, then $B \supseteq co_R(A)$.

See [11, pp.105-107] for the properties of R-hull. It can be shown that

$$co_R(A) = \cap\{B_R^c : B_R \cap A = \emptyset\}.$$

The R-hull of a closed set A may not exist, see [11, Remark 4.9]. However

PROPOSITION 2.4 *[11, Theorem 4.8] If A is a plane closed connected subset of an open circle of radius R, then A has R-hull.*

3. Properties of R-curves

In this section the definition and some properties of R-curves in $\mathbb{R}^n$ are introduced and proved.

Let $\gamma \subset \mathbb{R}^n$ be an oriented rectifiable curve and let $x(\cdot)$ be its parametric representation with respect to the arc length parameter $s \in [0, L]$. If $x_1 = x(s_1), x_2 = x(s_2) \in \gamma$ with $s_1 \leq s_2$, the notation $x_1 \preceq x_2$ will be used. Let us denote $x(s) = x$,

$$\gamma_x = \{y \in \gamma : y \preceq x\}; \quad \gamma_{x_1,x_2} = \{y \in \gamma : x_1 \preceq y \preceq x_2\}; \quad |\gamma_x| = \mathrm{length}(\gamma_x). \tag{3}$$

In this paper a *curve* in $\mathbb{R}^n$ is also the image of a continuous function on an interval, valued into $\mathbb{R}^n$.

Definition 4 Let R be a fixed positive number. An R-curve $\gamma \subset \mathbb{R}^n$ is a rectifiable oriented curve with arc length parameter $s \in [0, L]$, tangent vector $\mathbf{t}(s) = x'(s)$ such that the inequality

$$|x(s_1) - x(s) - R\,\mathbf{t}(s)| \geq R \tag{4}$$

holds for almost all s and for $0 \leq s_1 \leq s \leq L$.
Γ_R will denote the class of R-curves in $\mathbb{R}^n$.

The geometric meaning of (4) is that for every point $x = x(s) \in \gamma$, with tangent vector $\mathbf{t}(s)$, the set γ_x is outside of the open ball of radius R through x centered at $x + R\,\mathbf{t}(s)$.

Let us notice the following equivalent formulations of (4) for $0 \leq s_1 < s \leq L$:

$$|x(s_1) - x(s)|^2 - 2R\,\langle x(s_1) - x(s)), \mathbf{t}(s)\rangle \geq 0; \tag{5}$$

$$\langle x(s) - x(s_1), \mathbf{t}(s)\rangle \geq -\frac{|x(s_1) - x(s)|^2}{2R}; \tag{6}$$

$$\langle \frac{x(s_1) - x(s)}{|x(s_1) - x(s)|}, \mathbf{t}(s)\rangle \leq \frac{|x(s_1) - x(s)|}{2R}, \quad \text{if} \quad x(s_1) \neq x(s). \tag{7}$$

Lemma 3.1 *Let γ be a rectifiable curve with arc length parametrization $x(s)$. Then, the inequality* (4) *is equivalent to*

$$|x(s) - x(s_1)| \geq |x(s_2) - x(s_1)|e^{(s_2 - s)/(2R)} \quad \textit{for} \quad 0 \leq s_1 \leq s_2 \leq s \leq L. \tag{8}$$

Proof. Inequality (6) can be written as

$$\begin{gathered}
\tfrac{d}{ds}|x(s) - x(s_1)|^2 \geq -\tfrac{|x(s_1) - x(s)|^2}{R}, \\
\tfrac{d}{ds}\log|x(s) - x(s_1)|^2 \geq -\tfrac{1}{R}, \\
\tfrac{d}{ds}\big(\log|x(s) - x(s_1)|^2 + \tfrac{1}{R}(s - s_1)\big) \geq 0. \\
\log|x(s) - x(s_1)|^2 + \tfrac{1}{R}(s - s_1) \geq \log|x(s_2) - x(s_1)|^2 + \tfrac{1}{R}(s_2 - s_1), \\
|x(s) - x(s_1)|e^{\frac{s - s_1}{2R}} \geq |x(s_2) - x(s_1)|e^{\frac{s_2 - s_1}{2R}}.
\end{gathered}$$

Therefore (8) follows. Conversely, if γ satisfies (8), then inequality (6) is obtained, by using the previous inequalities in ascending order. □

Corollary 3.2 *An R-curve does not intersect itself.*

Proof. By contradiction, let us assume that $x(s) = x(s_1)$, $s_1 < s$. Then (8) implies that $x(s_2) = x(s) = x(s_1)$ for all s_2 between s_1 and s; so the curve $x(\cdot)$ is constant in $[s_1, s]$. This is impossible as s is the arc length. □

Remark 1 Let us notice that $\Gamma \subset \Gamma_R$, where Γ is the class of rectifiable curves γ for which for almost every $x \in \gamma$ with tangent vector $\mathbf{t}_x$ the set γ_x is contained in the half-plane bounded by the line through x, orthogonal to $\mathbf{t}_x$. Therefore if $\gamma \in \Gamma$, then (8) holds for every $R > 0$, namely

$$|x(s) - x(s_1)| \geq |x(s_2) - x(s_1)| \quad \text{for} \quad 0 \leq s_1 \leq s_2 \leq s \leq L. \tag{9}$$

This property is a "key" property of the curves of Γ (called self expanding property, see [3], [6]). Let us notice that the property (8), which seems an expanding property, makes the arc length of the R-curve appear, in striking difference from (9).

Theorem 3.3 *Let $\gamma \in \Gamma_R$. For every $s \in (0, L)$, $x = x(s)$, $\gamma_x \subsetneq \gamma$, the following two subsets of S^{n-1}:*

$$U_x^+ = \{u \in S^{n-1} : \exists s_k \geq s, \lim_{s_k \to s} x'(s_k) = u\}, \quad U_x^- = \{u \in S^{n-1} : \exists s_k \leq s, \lim_{s_k \to s} x'(s_k) = u\}$$

are non empty.
Moreover the following properties hold.
(i) if $x(\cdot)$ is differentiable at s, then $x'(s) \in U_x^+ \cup U_x^-$;
(ii) if $u \in U_x^+ \cup U_x^-$ then

$$|x(s_1) - x(s)|^2 - 2R\,\langle x(s_1) - x(s), u\rangle \geq 0 \quad \textit{for} \quad 0 \leq s_1 < s < L; \tag{10}$$

(iii) let $B^0 = B(x + Ru)$, $u \in S^1$ so that $B^0 \cap \gamma = \emptyset$, then

$$\exists u^+ \in U_x^+ : \langle u^+, u\rangle \leq 0, \qquad \exists u^- \in U_x^- : \langle u^-, u\rangle \geq 0; \tag{11}$$

(iv) if there exist $S^1 \ni u_k \to u$, $s_k \to s$, $s_k < s$, with $x(s_k) \in \partial B^k := \partial B(x + Ru_k)$, then

$$\exists u_1^- \in U_x^- : \langle u_1^-, u\rangle \leq 0. \tag{12}$$

Proof. The set $G = \{\tau : \tau \geq s : \exists\, x'(\tau)\}$ is a dense set in (s, L); let $\{\tau_k\} \subset G$ having s has an adherence point; by possibly passing to a subsequence, $\{x'(\tau_k)\}$ has limit $u \in S^{n-1} \in U_x^+$. Then U_x^+ and similarly U_x^- are non empty. Moreover as the sets $\gamma_{x(\tau_k)}$ have an R-support ball at $x(\tau_k)$, centered at $x(\tau_k) + Rx'(\tau_k)$, then (10) is obtained from (5) with $s = \tau_k$ passing to the limit.

To prove (11), let us notice that the assumption $B^0 \cap \gamma = \emptyset$ implies (4) (thus (6)) with u instead of $\mathbf{t}(s)$ and $s - \frac{1}{k}$ in place of s_1; therefore

$$\int_{s-\frac{1}{k}}^{s} \langle x'(t), u\rangle dt = \langle x(s) - x(s - \frac{1}{k}), u\rangle \geq -\frac{|x(s - \frac{1}{k}) - x(s)|^2}{2R} \quad \text{for} \quad s - \frac{1}{k} \in (0, L). \tag{13}$$

Then, for all k, sufficiently large, there exists $\sigma_k \in (s - \frac{1}{k}, s)$ with

$$\langle x'(\sigma_k), u\rangle \geq -k\frac{|x(s - \frac{1}{k}) - x(s)|^2}{R}.$$

Since by definition of arc length $|x(s - \frac{1}{k}) - x(s)| \leq \frac{1}{k}$, it follows that

$$\langle x'(\sigma_k), u\rangle \geq -\frac{|x(s - \frac{1}{k}) - x(s)|}{R}.$$

By possibly passing to a subsequence, $x'(\sigma_k) \to u^- \in U_x^-$; then the second inequality in (11) is proved as $|x(s - \frac{1}{k}) - x(s)| \to 0$. To prove first inequality in (11) let us notice that assumption $B^0 \cap \gamma = \emptyset$ implies (4) (thus (6)) with u instead of $\mathbf{t}(s)$ and $s_1 = s + \frac{1}{k} > s$, k large enough. Then

$$\int_{s}^{s+\frac{1}{k}} \langle x'(t), u\rangle dt = \langle x(s + \frac{1}{k}) - x(s), u\rangle \leq \frac{|x(s + \frac{1}{k}) - x(s)|^2}{2R} \quad \text{for} \quad s + \frac{1}{k} \in (0, L).$$

Arguing as above the first inequality in (11) is obtained.

To prove (iv), since $x(s_k) \in \partial B^k$, then equality holds in (13) with s_k in place of $s - \frac{1}{k}$ and u_k in place of u. Thus

$$\int_{s_k}^{s} \langle x'(t), u_k\rangle dt = -\frac{|x(s_k) - x(s)|^2}{2R} < 0.$$

Therefore there exists $\sigma_k \in (s_k, s)$ so that $\langle x'(\sigma_k), u_k\rangle \leq 0$ and $x'(\sigma_k) \to u_1^- \in U_x^-$. Then (12) is obtained by passing to the limit. □

4. *R*-curves in disks of radius *R*

In this section let us assume that γ is a plane R-curve of length $L = |\gamma|$ contained in a closed circle of radius less than R. Let $x(\cdot)$ be the parametric representation of γ with respect to the arc length and let $0 \leq s \leq L$, $x = x(s) \in \gamma$. According to Proposition 2.4, γ_x has R-hull $co_R(\gamma_x)$.

For a plane convex body K let us denote $per(K)$ the perimeter of the boundary ∂K. Let p be a point not in K. The simple cap body K^p is the convex hull of $K \cup \{p\}$, see [12].

For a vector $u = (a, b)$, let $u^\perp = (-b, a)$.

Theorem 4.1 *Let $x \in \gamma \in \Gamma_R$, $\gamma \subset B_R$. Let*

$$W_x = \{u \in S^1 : (B(x + Ru))^c \supset \gamma_x\}. \tag{14}$$

Then

$$U_x^+ \cup U_x^- \subset W_x; \tag{15}$$

moreover

$$W_x = \mathrm{Nor}_{\,co_R(\gamma_x)}(x) \cap S^1. \tag{16}$$

Proof. Let $u \in U_x^+ \cup U_x^-$, then inequality (10) holds, which means that $(B(x+Ru))^c \supset \gamma_x$, and (15) is proved. Let $w \in W_x$ and let $B = B(x+Rw)$. As $\gamma_x \subset B^c$ and $reach(B^c) = R$, then by (ii) of Definition 3, $co_R(\gamma_x) \subset B^c$. This fact implies that

$$\text{Nor}_{\,co_R(\gamma_x)}(x) \supseteq \text{Nor}_{\,B^c}(x) = \{\lambda w, \lambda \geq 0\}.$$

Then

$$\text{Nor}_{\,co_R(\gamma_x)}(x) \cap S^1 \supseteq W_x. \tag{17}$$

Let us prove now the opposite inclusion. Let $u \notin W_x$, $u \in S^1$. Then in $B(x+Ru)$ there are points of γ_x, thus there exists a point $y \in co_R(\gamma_x) \cap B(x+Ru)$, $y \neq x$. Then, by Proposition 2.3, $\mathfrak{h}(x,y,R) \cap co_R(\gamma_x)$ has a connected component F joining x and y, $F \subset (B(x+Ru) \cup \{x\})$. Therefore, as $\mathfrak{h}(x,y,R) \subset B(x+Ru) \cup \{x\}$, in the tangent cone of F at x there is a vector making an acute angle with u; so in the tangent cone to $co_R(\gamma_x)$ there is a vector making an acute angle with u. Then $u \notin \text{Nor}_{\,co_R(\gamma_x)}(x)$, the opposite inclusion of (17) holds and (16) is proved. □

Definition 5 Let b and c two distinct points in the plane with $|b-c| < 2R$. Let us consider the following geometric construction. Let $B(b)$ and $B(c)$ two open circles, of radius R and center b, c respectively. Let $x \in \partial B(b) \cap \partial B(c)$. Let l be the line through b and c, let H be the half plane with boundary l containing x.

The unbounded region $a\check{n}g(bxc) :\equiv B(b)^c \cap B(c)^c \cap H$ will be called a *curved angle*. Moreover

$$\text{meas}(a\check{n}g(bxc)) := \text{meas}(\text{Tan}_{\,a\check{n}g(bxc)}(x) \cap S^1)$$

is the measure of the angle between the half tangent lines at x to the boundary of $a\check{n}g(bxc)$.

It is not difficult to see that

$$\text{meas}(a\check{n}g(bxc)) = \pi - 2\arcsin\frac{|b-c|}{2R}. \tag{18}$$

When x, y are points on a circumference ∂B, let us denote with $arc(x,y)$ the shorter arc on ∂B from x to y.

Lemma 4.2 *Let $x, x_2 \in \mathbb{R}^2$, $|x-x_2| < R$. Let $B^2 = B(b,R)$ with $\partial B^2 \supset \{x, x_2\}$. Let $B^* = B(c_*)$ the ball of radius R, with ∂B^* orthogonal at x_2 to ∂B^2 and $x \in B^*$. Let us assume that there exists $x_1 \in (B^* \cup B^2)^c$ with the properties:*
(i) $|x_1 - x| < R, |x_2 - x_1| < R$;
(ii) x_1 lies in the half plane with boundary the line through x and x_2 not containing b;
(iii) there exists $B^1 = B(c_1, R)$ with $\{x_1, x\} \subset \partial B^1$, with $arc(x, x_1) \subset (B^2)^c$, such that the line through x and x_1 separates c_1 and x_2.

Then the measure of the curved angle $a\check{n}g(bxc_1)$ is less than $\pi/2$.

Proof. Let $\{w\} = arc(x, x_1) \cap \partial B^*$(possibly it can be $w = x_1$). As $|x-w| < R$, $|x-b| = R$, then

$$|w-b| < |x-w| + |x-b| < R + R = 2R.$$

Let us remark that the circles B^* and B^1 both have w on the boundary (see Fig.1).

It is not difficult to see that the convex angle $b\hat{w}c_1$ contains the convex angle $b\hat{w}c_*$. The two triangles bwc_1 and bwc_* have one side in common and the sides wc_*, wc_1 have the same length R. Then $|b-c_*| < |b-c_1|$. Thus by (18)

$$\pi/2 = \text{meas}(a\check{n}g(bx_2c_*) > \text{meas}(a\check{n}g(bxc_1) \tag{19}$$

and the thesis is proved. □

Theorem 4.3 *Let $N > 1$. Let γ be a C^1 plane R-curve. Assume that for every $x \in \gamma$, γ_x is contained in the disk $D(x, R/N)$. Then, the measure of* $\text{Nor}_{\,co_R(\gamma_x)}(x) \cap S^1$ *is equal or greater than $\pi/2$.*

Proof. Let $H_i = (B(x+Ru_i))^c, i = 1,2$, where u_i are the two vectors bounding $W_x = S^1 \cap \text{Nor}_{\,co_R(\gamma_x)}(x)$ (may be $u_1 = u_2$ then $H_1 = H_2$). Then

$$co_R(\gamma_x) \subset H_1 \cap H_2 \cap D(x, R/N).$$

If $u_1 = -u_2$ then the measure of $\text{Nor}_{\,co_R(\gamma_x)}(x)$ is equal π and the thesis holds.

Let $u_1 \neq -u_2$.

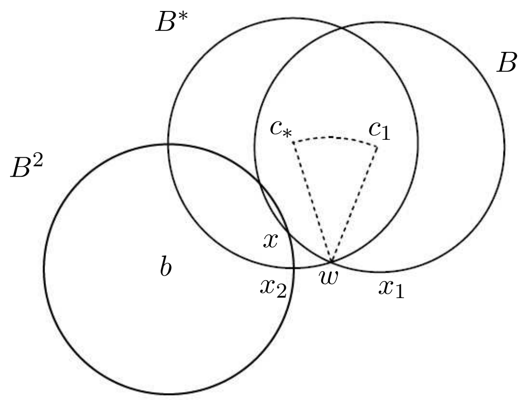

Figure 1. Curved angles

There are two possible cases:
(a) at least one of the two sets $\gamma_x \cap \partial H_i \setminus \{x\}$, i=1,2, is empty;
(b) there exist at least two points $x_i \in \gamma_x \cap \partial H_i, x_i \neq x$ (i=1,2).

Case (a): with no loss of generality one can assume that

$$\gamma_x \cap \partial H_1 \setminus \{x\} = \emptyset.$$

Let $u_1 = (\cos\alpha_1, \sin\alpha_1), 0 \leq \alpha_1 < 2\pi$ and let

$$u^\delta := (\cos(\alpha_1 - \delta), \sin(\alpha_1 - \delta)).$$

By definition of the vectors u_1, u_2, bounding $\mathrm{Nor}_{co_R(\gamma_x)}(x) \cap S^1$, for $\delta > 0$ sufficiently small, one has

$$\gamma_x \cap B(x + Ru^\delta) \neq \emptyset.$$

This means that, for $\delta = \frac{1}{k} > 0$ and k sufficiently large, there exists a sequence $s_k \to s$, $s_k < s$ such that

$$x(s_k) \in \partial B(x + Ru^{\frac{1}{k}}) \setminus D(x + Ru_1, R).$$

Then, by (iv) of Theorem 3.3, there exists $u_1^- \in U_x^- \subset \mathrm{Nor}_{co_R(\gamma_x)}(x)$ so that $\langle u_1^-, u_1 \rangle \leq 0$. The thesis follows.

Case (b): with no restriction one can assume $x_i = x(s_i)$ (i=1,2), with $s_1 < s_2 < s$ and that the triangle $x_1 x_2 x$ is clockwise oriented, see Fig.1.
Let us consider the point $x_2 \in \partial B(x + Ru_2)$. Let $\tilde{u_2}$ so that $x + Ru_2 = x_2 + R\tilde{u_2}$ and let

$$B^2 := B(x + Ru_2) = B(x_2 + R\tilde{u_2}), \quad B^1 := B(x + Ru_1).$$

Let us notice that, by Theorem 4.1, $\tilde{u_2} \in W_{x_2} = S^1 \cap \mathrm{Nor}_{co_R(\gamma_{x_1,x_2})}(x_2)$. As $\gamma_{x_2,x} \subset (B^2)^c$ the tangent vector $x'(s_2)$ is tangent to ∂B^2 at $x_2 = x(s_2)$. That is $\langle x'(s_2), \tilde{u_2} \rangle = 0$.

Let us consider the closed region Q bounded by $arc(x_2, x)$ on ∂B^2, $arc(x, x_1)$ on ∂B^1 and γ_{x_1,x_2}. Let us show that as $\gamma_x \subset H_1 \cap H_2 \cap D(x, R/N)$, then $\gamma_{x_2,x} \subset Q$. Otherwise, $\gamma_{x_2,x}$ would have a point $y \neq x_2$ outside Q and it would be $\gamma_{x_1,x_2} \cap \gamma_{y,x} \neq \emptyset$, in contradiction with Corollary 3.2. Then $x'(s_2)$ is tangent to $arc(x_2, x)$ at x_2 and $\langle x'(s_2), -\tilde{u_2}^\perp \rangle = 1$.

Let $B^* := B(x_2 + Rx'(s_2))$. As $x'(s_2) \in U_{x_2}^+ \cup U_{x_2}^-$, by (15) the following inclusion holds

$$\gamma_{x_1,x_2} \subset (B^*)^c.$$

Then the point $x_1 \notin B^*$ and, as $|x_2 - x| < R$, the ball B^* contains x. Let $c_1 = x + Ru_1$, $b = x_2 + R\tilde{u_2}$, c_* the center of B^*.

Let us consider the curved angles $\check{ang}(bxc_1)$ and $\check{ang}(bx_2c_*)$. As ∂B^2 and ∂B^* are orthogonal at x_2, the hypotheses of Lemma 4.2 for x, x_2, b, c_1, x_1 are satisfied. Thus

$$\text{meas}(\check{ang}(bxc_1)) < \pi/2.$$

Since

$$\text{meas}(\text{Nor}_{co_R(\gamma_x)}(x) \cap S^1) = \pi - \text{meas}(\check{ang}(bxc_1))$$

the thesis follows. ∎

Remark 2 The assumption on the regularity of γ in Theorem 4.3 will be removed in a work in progress. Previous theorem provides a bound for the measure of $\text{Nor}_{co_R(\gamma_x)}(x)$ for R-curves γ in a small circle. This bound implies, Theorem 4.4, a bound on the measure of the tangent angle $\text{Tan}_{co(\gamma_x)}(x)$.

Definition 6 Let γ be an oriented curve, $x \in \gamma$, $N > 1$; γ satisfies the property $P_N(\gamma_x)$ if

$$\text{meas}(\text{Tan}_{co(\gamma_x)}(x)) \leq \pi/2 + 2\arcsin\frac{1}{2N} \tag{20}$$

holds.

THEOREM 4.4 *Let γ be a C^1 plane R-curve. Assume that for every $x \in \gamma$, γ_x is contained in $D(x, R/N)$, $N > 1$. Then γ_x satisfies $P_N(\gamma_x)$ for every $x \in \gamma$.*

Proof. Let u_1, u_2 be, as in the previous theorem, the unit vectors bounding $S^1 \cap \text{Nor}_{co_R(\gamma_x)}(x)$. If $u_1 = -u_2$, then γ_x is contained in an equilateral triangle with vertex x, then $\text{Tan}_{co(\gamma_x)}(x)$ is acute and the thesis holds. If $u_1 \neq -u_2$ let A_1 the connected component of

$$(B(x + Ru_1))^c \cap (B(x + Ru_2))^c \cap D(x, R/N),$$

containing x. Let us notice that $reach(A_1, x) \geq R$ and $\text{Nor}_{co_R(\gamma_x)}(x) = \text{Nor}_{A_1}(x)$. Then, by Proposition 2.1, the sets $\text{Tan}_{A_1}(x)$ and $\text{Tan}_{co_R(\gamma_x)}(x)$ coincide. Then, by Theorem 4.3,

$$\text{meas}(\text{Tan}_{A_1}(x)) \leq \pi/2.$$

It is an easy exercise to show that

$$\text{meas}(\text{Tan}_{co(A_1)}(x)) = \text{meas}(\text{Tan}_{A_1}(x)) + 2\arcsin\frac{1}{2N}.$$

As $co(\gamma_x) \subset co(A_1)$,

$$\text{meas}(\text{Tan}_{co(\gamma_x)}(x)) \leq \text{meas}(\text{Tan}_{co(A_1)}(x)),$$

from the previous equality and inequalities, the property $P_N(\gamma_x)$ follows. ∎

Remark 3 This theorem proves that the R-curves, satisfying the assumptions of Theorem 4.3, are ϕ-self-approaching curves with $\phi = \pi/2 + 2\arcsin\frac{1}{2N}$, opposite oriented, according to Definition 1 in [7].

In what follows, bounds for the curves' length and detour are proved with a simple extension of the techniques in [1] and in a different way than [7].

Let, for simplicity, $|\gamma|$ be the length of γ, $\gamma(s) = \gamma_{x(s)}$ and $p(s) := per(co(\gamma(s)))$.

THEOREM 4.5 *Let R be a positive number and let $N > 1$. Let z_0 be a fixed point in the plane. If γ is a plane R-curve, $\gamma \subset D(z_0, R/(2N))$ and the property $P_N(\gamma_x)$ holds for every $x \in \gamma$, then*

$$p'(s) \geq 1 - \frac{1}{N} \quad a.e. \quad s \in [0, |\gamma|]; \tag{21}$$

$$|\gamma| \leq \frac{\pi}{N-1}R. \tag{22}$$

Lemma 4.6 *Let K be a plane convex body. Let $p_0 \in \partial K$, $u = (\cos\alpha, \sin\alpha) \in -\mathrm{Tan}_K(p_0)$. Let $0 \le \omega < \pi$ the amplitude of $\mathrm{Tan}_K(p_0)$. Let $\varepsilon > 0$, $p_\varepsilon = p_0 + \varepsilon u$, K^{p_ε} the simple cap body of K at p_ε, then*

$$per(K^{p_\varepsilon}) - per(K) \ge \varepsilon(1 + \cos\omega). \tag{23}$$

Proof. Let $T := \mathrm{Tan}_K(p_0)$ and $N := \mathrm{Nor}_K(p_0)$ the normal cone of K at p_0. Since the amplitude of T is less than π, $u \notin T$. The assumptions of [6, Theorem 3.1] hold and formula [6, (19)] implies that

$$per(K^{p_\varepsilon}) - per(K) \ge \varepsilon \int_{N \cap S^1 \cap \{u\}^*} \langle \Theta, u \rangle d\Theta,$$

where $\Theta = (\cos\theta, \sin\theta)$. Since $u \in -T$, then $N = (-T)^* \subset \{u\}^*$ and the amplitude of N is $\pi - \omega$. Let $N \cap S^1 = \{(\cos\theta, \sin\theta), 0 \le \theta \le \pi - \omega\}$. The previous integral is equal to

$$\int_0^{\pi-\omega} \cos(\theta - \alpha) d\theta = \sin(\omega + \alpha) + \sin\alpha := f(\alpha).$$

Since $u \in -T$ the constraint $\frac{\pi}{2} - \omega \le \alpha \le \frac{\pi}{2}$ holds and in that interval $f(\alpha)$ is bounded below by $1 + \cos\omega$. $\square$

Proof. The proof is strongly similar to the proof of Theorem IV in [1]. First let us observe that $p(s)$, the perimeter of $co(\gamma(s))$, is increasing since $\gamma(s)$ is increasing by inclusion; thus $p(s)$ and $x(s)$ are derivable a.e in $[0, |\gamma|]$. Let us consider a point x on the curve γ, let us observe that as $\gamma \subset D(z_0, R/(2N))$, then for all $x \in \gamma$, $\gamma_x \subset D(x, R/N)$. By (20) the measure ω of $T := \mathrm{Tan}_{co(\gamma_x)}(x)$ satisfies

$$\omega \le \frac{\pi}{2} + 2 \arcsin \frac{1}{2N} < \pi. \tag{24}$$

By assumption $co(\gamma_x) \subset T \cap D(x, R/N)$ and for $h > 0$ the point $\overline{x} = x(s) + hx'(s)$ is in the angle opposite to T. Since ω is less than π, then $\overline{x} \notin T$. Let $co(\gamma(s))^{\overline{x}} :\equiv co(co(\gamma(s)) \cup \{\overline{x}\})$ the simple cap body of $co(\gamma(s))$ at $\overline{x}$. Let $per(K)$ be the perimeter of a plane convex body K. The hypothesis of Lemma 4.6 are satisfied with $K = co(\gamma(s))$, $p_0 = x(s) \in \partial K$, $u = x'(s)$, $\varepsilon = h$. Thus

$$per(co(\gamma(s))^{\overline{x}}) - per(co(\gamma(s)) \ge (1 + \cos\omega)h. \tag{25}$$

Let $w := x(s+h)$ and let us consider $co(\gamma(s))^w$ the simple cap body of $co(\gamma(s))$ at w. Arguing as in the proof of [1, Theorem VII, p. 222, line 11], the following asymptotic inequality holds:

$$per(co(\gamma(s))^{\overline{x}}) \ge per(co(\gamma(s))^w) + o(h), \quad \text{for} \quad h \to 0^+. \tag{26}$$

Since

$$co(\gamma(s)) \subset co(\gamma(s))^w \subset co(\gamma(s+h)),$$

then

$$p(s) \le per(co(\gamma(s))^w) \le p(s+h).$$

From (26):

$$p(s+h) - p(s) \ge per(co(\gamma(s))^w) - per(co(\gamma(s))) \ge per(co(\gamma(s))^{\overline{x}}) - per(co(\gamma(s))) + o(h)$$

and from (25)

$$p(s+h) - p(s) \ge (1 + \cos\omega)h + o(h) \quad \text{for} \quad h \to 0^+.$$

Thus, from (24),

$$p'(s) \ge 1 + \cos\omega \ge 1 - \frac{1}{N}, \quad \text{a.e.} \quad s \in [0, |\gamma|].$$

This proves (21). As $p(s)$ is a not decreasing functions, by integrating (21) in $[0, |\gamma|]$, the following inequality

$$p(|\gamma|) - p(0) \geq (1 - \frac{1}{N})|\gamma| \tag{27}$$

holds. As $co(\gamma)$ is contained in a circle of radius $\frac{R}{2N}$ then $p(|\gamma|) \leq \pi \frac{R}{N}$ and (22) is proved. □

THEOREM 4.7 *Let γ be a plane R-curve, contained in a circle of radius less than R/M and centered at z_0, with $M > 2$. Let $P_M(\gamma_x)$ holds for every $x \in \gamma$. Then the detour of $\gamma_{x_1,x}$ is bounded by a constant $c(M)$. Moreover if $M \geq 3$, $c(M) \leq 6\pi e^{\pi}$.*

Proof. From (8) of Lemma 3.1, for $0 \leq s_1 \leq s_2 \leq s \leq L$

$$|x(s) - x(s_1)|e^{\frac{|\gamma|}{2R}} \geq |x(s_2) - x(s_1)|.$$

Therefore the circle of radius $|x(s) - x(s_1)|e^{\frac{|\gamma|}{2R}}$ centered in $x(s_1)$ contains $\gamma_{x(s_1),x(s)}$. It follows that

$$per(co(\gamma_{x(s_1),x(s)})) \leq 2\pi|x(s) - x(s_1)|e^{\frac{|\gamma|}{2R}}. \tag{28}$$

Let $x_1 = x(s_1), x = x(s)$; by assumption $\gamma_{x_1,x} \subset D(x_0, R/M)$; then, from (27) of Theorem 4.5 with $N = M/2$, it follows that

$$\frac{|\gamma_{x_1,x}|}{per(co(\gamma_{x_1,x}))} \leq \frac{1}{1 - \frac{2}{M}} = \frac{M}{M-2};$$

then, from (28)

$$\frac{|\gamma_{x_1,x}|}{|x - x_1|} = \frac{|\gamma_{x_1,x}|}{per(co(\gamma_{x_1,x}))} \frac{per(co(\gamma_{x_1,x}))}{|x_1 - x|} \leq \frac{M}{M-2} 2\pi e^{\frac{|\gamma|}{2R}}.$$

From (22), it follows that

$$\frac{|\gamma_{x_1,x}|}{|x - x_1|} \leq 2\pi \frac{M}{M-2} e^{\frac{\pi}{(M-2)}}.$$

Then $c(M) \leq 2\pi \frac{M}{M-2} e^{\frac{\pi}{(M-2)}}$. If $M \geq 3$, then $c(M) \leq 6\pi e^{\pi}$. □

Remark 4 The bound for $c(M)$ in the previous theorem is not sharp. A better bound can be obtained using [7, Theorem 7].

5. Bounds for the length and the detour of plane *R*-curves

LEMMA 5.1 *Let $0 < r_1 < \tau$ and let $x_0, \ldots, x_m$ be points in the closed ball $D(w_0, \tau)$ of $\mathbb{R}^n$, satisfying*

$$|x_i - x_j| \geq r_1, \quad \textit{for} \quad 0 \leq i \neq j \leq m.$$

Then

$$m \leq (\frac{4\sqrt{n}\tau}{r_1})^n. \tag{29}$$

Proof. The cubes Q_j centered in x_j with sides $r_1/\sqrt{n}$ do not have internal points in common; moreover each cube Q_j is contained in the cube Q centered in x_0 with side 4τ. Since

$$\sum_j \text{meas}(Q_j) \leq \text{meas}(Q)$$

the bound (29) is obtained. □

The following theorem gives an answer to question (B) of the introduction.

THEOREM 5.2 *Let $w_0 \in \mathbb{R}^2$, $R > 0$. Let γ be a C^1 plane R-curve, $\gamma \subset D(w_0, \tau)$. Then there exists a positive constant $c(R, \tau)$, depending on R and τ only so that*

$$|\gamma| \leq c(R, \tau), \tag{30}$$

where
(i) if $\tau \leq \frac{R}{4}$, then $c(R, \tau) \leq 4\pi\tau \leq \pi R$;
(ii) if $\tau > \frac{R}{4}$ then

$$c(R, \tau) \leq (1 + (16\sqrt{2}e^{\pi/2})^2 (\frac{\tau}{R})^2)\pi R. \tag{31}$$

Proof. Case (i): let $N = \frac{R}{2\tau} \geq 2$. Then

$$\gamma \subset D(w_0, \tau) = D(w_0, \frac{R}{2N}).$$

Then, by Theorem 4.5,

$$|\gamma| \leq \frac{\pi}{N-1} R = \frac{\pi}{\frac{R}{2\tau} - 1} R = 2\pi(1 + \frac{2\tau}{R - 2\tau})\tau.$$

As $R \geq 4\tau$, $|\gamma| \leq 4\pi\tau$ and in case (i) inequality (30) holds.
Case (ii): let γ_0 be the closed connected component of $\gamma \cap D(x(0), R/4)$ starting at $x(0)$, then Theorem 4.5 applies to γ_0 with $z_0 = x(0)$, $N = 2$; by (22), it follows that $|\gamma_0| \leq \pi R$. If $\gamma_0 \cap \partial B(x(0), R/4) = \emptyset$, thus $\gamma = \gamma_0$ and by previous inequality $|\gamma| \leq \pi R$. Thus (30) is proved with the constant given in (31).

In case $E_0 := \gamma_0 \cap \partial B(x(0), R/4) \neq \emptyset$. Let x_1 be the end point of γ_0. Let γ_1 be the closed connected component of $(\{x_1\} \cup (\gamma \setminus \gamma_{x_1})) \cap D(x_1, R/4)$. Then, by Theorem 4.5

$$|\gamma_1| \leq \pi R.$$

Let $\gamma_1 \cap \partial B(x_1, R/4) = \emptyset$, thus $\gamma = \gamma_0 \cup \gamma_1$; then

$$|\gamma| = |\gamma_0| + |\gamma_1| \leq 2\pi R$$

and (30) is proved with the constant given in (31).

Let us assume that $E_1 := \gamma_1 \cap \partial B(x_1, R/4) \neq \emptyset$. An iterative procedure can be constructed. Let us assume that $\gamma_0, \gamma_1, \ldots, \gamma_m$ are connected subsets of γ already defined; let x_j and x_{j+1} be the starting and the end points of each γ_j $(j = 0, \ldots, m-1)$; γ_j is the closed connected component of $\big(\{x_j\} \cup (\gamma \setminus \gamma_{x_j})\big) \cap D(x_j, R/4)$ starting at x_j; moreover $\gamma_j \cap \partial B(x_j, R/4) \neq \emptyset$ $(j = 0, \ldots, m-1)$ and

$$|\gamma_j| \leq \pi R. \tag{32}$$

Let us consider

$$E_m := (\{x_m\} \cup (\gamma \setminus \gamma_{x_m})) \cap \partial(B(x_m, R/4)).$$

There are two possibilities: either $E_m = \emptyset$ or $E_m \neq \emptyset$. If $E_m = \emptyset$ then the procedure stops and $\gamma = \cup_{i=0}^m \gamma_i$. Otherwise, if $E_m \neq \emptyset$, let x_{m+1} be the end point of γ_m and let γ_{m+1} be the closed connected component of $\big(\{x_{m+1}\} \cup (\gamma \setminus \gamma_{x_{m+1}})\big) \cap D(x_{m+1}, R/4)$. If γ_{m+1} reduces to the point x_{m+1} the procedure stops. Otherwise the procedure continues.
Claim : $|x_i - x_j| \geq \frac{R}{4} e^{-\frac{\pi}{2}}$ for $0 \leq i \neq j \leq m$.
The claim will be proved later.
From Lemma 5.1 with $r_1 := \frac{R}{4} e^{-\frac{\pi}{2}}$, since $\{x_0, \ldots, x_m\} \subset \gamma \subset D(w_0, \tau)$, the iterative procedure stops with $m \leq (\frac{4\sqrt{2}\tau}{r_1})^2$. Then $\gamma = \cup_{i=0}^m \gamma_i$; from (32) and the previous bound on m it follows that

$$|\gamma| = \sum_{i=0}^m |\gamma_i| \leq (m+1)\pi R \leq \big(1 + (\frac{4\sqrt{2}}{r_1})^2 \tau^2\big)\pi R.$$

Inequality (30) follows with $c(R, \tau)$ given by (31).

Proof of the claim: Let $x_i = x(s_i), x_j = x(s_j), s_i < s_j$. The claim holds true if $x_j \notin B(x_i, R/4)$. Assume that $x_j \in B(x_i, R/4)$. Let us recall that $\gamma \setminus \gamma_{x_j}$ has points outside of $B(x_i, R/4)$. Thus the connected component $\overline{\gamma}$ of $\gamma \cap D(x_i, R/4)$ that contains x_j reenters in $D(x_i, R/4)$ in a point $x(\overline{s}) \in \partial B(x_i, R/4)$, $s_i < \overline{s} < s_j$. By Theorem 4.5

$$s_j - \overline{s} \leq |\overline{\gamma}| < \pi R.$$

By Lemma 3.1, as $0 \leq s_i < \overline{s} < s_j$ then

$$|x(s_j) - x(s_i)| \geq |x(\overline{s}) - x(s_i)| e^{-\frac{s_j - \overline{s}}{2R}} \geq \frac{R}{4} e^{-|\overline{\gamma}|/2R} \geq \frac{R}{4} e^{-\pi/2}$$

holds. The claim is proved. □

The bound for the constant $c(R, \tau)$ obtained above is not the best one, but the exponent of the factor τ^2 in (31) cannot be lowered. As an example, let us consider the square of sides $(p+1)R$, $p \in \mathbb{N}$, p pair, with vertices $O = (0,0), (pR, 0), (pR, pR), (0, pR)$. Let γ the piecewise linear line joining the points

$$\begin{aligned}
&(0,0), (pR, 0), (pR, R), (0, R), \\
&\dots \\
&(0, 2kR), (pR, 2kR), (pR, (2k+1)R), (0, (2k+1)R), \\
&\dots \\
&(0, (p-2)R), (pR, (p-2)R), (pR, (p-1)R), (0, (p-1)R), \\
&(0, pR), (pR, pR), (pR, (p+1)R).
\end{aligned}$$

γ is a piecewise linear R-curve with length $(p+1)(pR) + pR + R = (1+p)^2 R$. Let $(p+1)\sqrt{2}R = \tau$. Then $\gamma \subset B(O, \tau)$ and $|\gamma| = \tau^2/2R$. With a standard smoothing technique the previous example can be extended to a C^1 plane R-curve.

THEOREM 5.3 *Let $z_0 \in \mathbb{R}^2$, $R > 0$. Let γ be a C^1 plane R-curve, $\gamma \subset D(z_0, \tau)$. Then the detour of $\gamma_{x_1,x}$ is bounded for all $x_1, x \in \gamma$ by a constant depending on R and τ only.*

Proof. Let $x_1, x \in \gamma$, $x_1 \prec x$. If $\gamma_{x_1,x} \subset D(x_1, R/3) \subset D(z_0, \tau)$ then the result follows from Theorem 4.7. Otherwise $per(co(\gamma_{x_1,x})) \geq 2|x - x_1| > \frac{2}{3}R$. Then

$$\frac{1}{|x - x_1|} < \frac{3}{R}.$$

Then from (30) and the previous inequality

$$\frac{|\gamma_{x_1,x}|}{|x - x_1|} < 3\frac{c(R,\tau)}{R}$$

follows. □

Let us conclude this section by showing that in the previous theorem the dependence on τ is needed.

PROPOSITION 5.4 *Let $x_0, \overline{x}$ be two given points with distance $2R$. For every $K > 0$ there exists $\gamma \in \Gamma_R$, with first point x_0 and last point $\overline{x}$ such that the detour*

$$\frac{|\gamma_{x_0,\overline{x}}|}{|x_0 - \overline{x}|} > K.$$

Proof. Let $m \geq 3$ be a real number. Let C a circumference of radius $\rho = mR$ through $x_0, \overline{x}$. Let γ be obtained from C by deleting the shorter arc joining $x_0, \overline{x}$. The curve γ is an R-curve, it satisfies (4) for every $x(s) \in \gamma$. The arc $\gamma_{x_0,\overline{x}}$ has detour

$$\frac{(2\pi - 2\arcsin\frac{R}{\rho})\rho}{2R} = (2\pi - 2\arcsin\frac{1}{m})\frac{m}{2} = (\pi - \arcsin\frac{1}{m})m.$$

This number can be made arbitrarily large, by choosing m suitably. □

6. R-curves as steepest descent curves

In this section the R-curves are seen as steepest descent curves of classes of functions. The bound on their length proved in previous sections, generalizes the results of [1], [2], [3], [4], [5], [6], for quasi convex functions.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected set. Ω will be called regular if for every $y \in \partial\Omega$ there exists a neighborhood U of y so that $\partial\Omega \cap cl(U)$ is a regular curve. Let Ω be regular, $u \in C^2(cl(\Omega))$, $Du(x) \neq 0$ for $u(x) > \min_{cl(\Omega)} u$. Let us consider the sublevel sets of u in $cl(\Omega)$: $\Omega_l = \{x \in cl(\Omega) : u(x) \leq l\}$, for $l > \min_{cl(\Omega)} u$ and let assume that $\partial\Omega = \Omega_{\max u}$. Let $\text{argmin}u := \{x : u(x) = \min_{cl(\Omega)} u\}$.

A simple rectifiable curve γ will be called regular if its parametric representation $x(\cdot)$ with respect to its arc length is C^2. Let us recall that γ (with ascent parameter s) is called a steepest descent curve for the function u in Ω if it is a solution of the differential equation

$$x' = \frac{Du(x)}{|Du(x)|} \quad x \in \Omega \setminus \text{argmin}u.$$

Theorem 6.1 *Let Ω, u be satisfying the above assumptions. Let $R > 0$. If all sublevel sets Ω_l of u have the property of the R-exterior ball, then*
(i) the steepest descent curves of u are R-curves,
(ii) their lengths are uniformly bounded by a constant depending only on R and the diameter of Ω.

Proof. The set Ω_l has the property of the R-exterior ball (Definition 2); then $\forall y \in \partial\Omega_l, \forall x \in \Omega$, such that $u(x) \leq u(y)$,

$$|x - (y + R\frac{Du(y)}{|Du(y)|})| \geq R \tag{33}$$

holds.

Let $x(\cdot)$ a steepest descent curve for u; let $y = x(s)$, $x = x(s-h)$ with $h > 0$, then $u(x) < u(y)$ and from (33), it follows that

$$|x(s-h) - (x(s) + R\frac{Du(x(s))}{|Du(x(s))|})| \geq R.$$

As $Du(x(s))/|Du(x(s))|$ is the tangent vector $\mathbf{t}(s)$ at $x(s)$, the inequality (4) holds and (i) is proved. The assert (ii) follows from Theorem 5.2. □

Theorem 6.2 *Let Ω and u be satisfying the assumptions of the previous theorem. Let $R > 0$ and let all the sublevel sets of u have reach greater than R; then*
(a) the steepest descent curves of u are R-curves;
(b) the steepest descent curves of u have length bounded by $c(R, \text{diam}\,\Omega)$.

Proof. Let $y \in \Omega \setminus \text{argmin}\, u$. The sets Ω_l have reach greater than R; then, by Proposition 2.2, Ω_l have the property of R exterior ball. Therefore, by (ii) of Theorem (6.1), the thesis holds with $c(R, \text{diam}\,(\Omega))$ given by (31). □

Next Theorem 6.6 provides a simple way to check when a "small" connected compact plane set A has the property of R-exterior ball. The goal of what follows is to prove that if $A \subset B_R$ and ∂A has the curvature equal or greater than $-1/R$ in each point, then A has the R-exterior ball property (Theorem 6.6).

Let η be the support of a plane oriented regular simple curve parametrized by $s \to x(s)$, s arc length. The signed curvature k_η at a point $x(s)$ is defined by the Frenet formula:

$$\frac{d}{ds}\mathbf{t}(s) = k_\eta \mathbf{n}(s),$$

where $\mathbf{t}(s)$ and $\mathbf{n}(s)$ are the tangent and the normal vector to η at $x(s)$; it is assumed that a counterclockwise rotation of $\pi/2$ maps $\mathbf{t}(s)$ on $\mathbf{n}(s)$. When η is the graph of a function $y = f(x)$, $f \in C^2(-L, L)$, oriented according to the x-axis orientation, the curvature of η at a point $(x, f(x))$ is

$$k_\eta = \frac{f''}{(1 + (f')^2)^{3/2}}(x).$$

Lemma 6.3 *Let $I = [0, l]$ ($[-l, 0]$), with $0 < l < R$. Let $f : I \to \mathbb{R}$ a C^2 real function. Let*

$$g(x) = \sqrt{R^2 - x^2} - R, \quad x \in I.$$

Let f satisfy the conditions:

$$f(0) = 0, \quad f'(0) = 0, \tag{34}$$

$$\frac{f''}{(1+(f')^2)^{3/2}}(x) \geq -\frac{1}{R}, \quad x \in I, \tag{35}$$

$$f(l) \leq g(l) \quad (f(-l) \leq g(-l)). \tag{36}$$

Then

$$f(x) \equiv g(x), \quad x \in I.$$

Proof. Let $I = [0, l]$. Since

$$\frac{g''}{(1+(g')^2)^{3/2}} = -\frac{1}{R},$$

inequality (35) implies that

$$\frac{d}{dx}\frac{f'}{(1+(f')^2)^{1/2}} \geq \frac{d}{dx}\frac{g'}{(1+(g')^2)^{1/2}}.$$

Thus, as $f'(0) = g'(0)$, integrating the previous inequality between 0 and x we obtain

$$\frac{f'}{(1+(f')^2)^{1/2}} \geq \frac{g'}{(1+(g')^2)^{1/2}}, \quad 0 \leq x \leq l,$$

As the function $\frac{t}{(1+t^2)^{1/2}}$ is strictly increasing in $t \in \mathbb{R}$, the inequality

$$f'(x) \geq g'(x), \quad 0 \leq x \leq l,$$

holds in $[0, l]$. As

$$0 \geq f(l) - g(l) = \int_0^l (f'(x) - g'(x))dx \geq 0,$$

then $f' \equiv g'$ in $[0, l]$. As $f(0) = g(0)$ then, $f \equiv g$ in $[0, l]$.

Let $I = [-l, 0]$. For the functions $\tilde{f} = f(-(\cdot))$ and $\tilde{g} = g$ the previous procedure applies in $[0, l]$, then the thesis follows. ∎

Lemma 6.4 *Let G, H be plane, open, bounded, simply connected sets with $G \subset H$. Let ∂G, ∂H have the same orientation. Assume that there exists $\overline{y} \in \partial G \cap \partial H$ and that in a neighborhood U of $\overline{y}$ the set $U \cap \partial G$ ($U \cap \partial H$) is support of a regular curve α (β) with orientation induced by ∂G (∂H). At $\overline{y}$ the sets G and H have the same exterior normal vector. Thus α and β have the same tangent vector at $\overline{y}$, accordingly to their orientation.*

Lemma 6.5 *Let B^i, the open disk of radius R, centered at $w^i \in \mathbb{R}^2, i = 0, 1$, $0 < |w^1 - w^2| < 2R$. Let η be an oriented regular plane curve joining two different points $y_0, y_1 \in \partial B^1$ such that*

$$\eta \subset cl(B^0) \setminus B^1.$$

Let's assume that one of the points y_0, y_1 is in B^0. Let y_1 follow y_0 according to the clockwise orientation of ∂B^1 and $y_0 \prec y_1$ on η. If the curvature of η satisfies the inequality

$$k_\eta \geq \frac{-1}{R}, \tag{37}$$

then

$$\eta \subset \partial B^1. \tag{38}$$

Proof. Let

$$B^t = B(tw_1 + (1-t)w_0, R), \quad 0 \le t \le 1$$

the family of plane balls connecting B^0 with B^1. Let $E = \{t : \eta \subset cl(B^t)\}$. By assumption, $0 \in E$. Let $t^* = \sup E$. If $t^* = 1$, the lemma is proved. Otherwise, since $\eta \subset cl(B^{t^*})$, there exists $\overline{y} \in \eta \cap \partial B^{t^*}$, $\overline{y} \neq y_0, y_1$. Thus $\eta \cap \partial B^{t^*}$ is a non empty set and of course it is closed. Let us prove that it is also open. At each point $\overline{y} \in \eta \cap \partial B^{t^*}$ let us consider a Cartesian coordinate system with origin $\overline{y}$ and x-axis oriented as the tangent to ∂B^{t^*} clockwise oriented. As $\eta \subset B^{t^*}$ and it is a regular curve, then η is tangent to ∂B^{t^*} at $\overline{y}$. In a neighborhood of $\overline{y}$ the support of η is the graph of a function $y = f(x)$ in the coordinate system (x, y). Let us prove that η is oriented accordingly to the graph of f, to say according to the x-axis orientation at $\overline{y}$. Let $H :\equiv B^{t^*} \setminus cl(B^1)$, G the open set bounded by η and $arc(y_0, y_1)$ on ∂B^1 clockwise oriented. Then $\overline{y}$ satisfies the assumptions of Remark 6.4 and the curves η and $\partial B^{t^*} \setminus B^1$ have the same tangent vector. Then the assumptions of Lemma 6.3 are satisfied in a suitable neighborhood of $\overline{y}$ since the bound (37) implies (35). Lemma 6.3 implies that $\eta \cap \partial B^{t^*}$ is also open. Then $\eta \cap \partial B^{t^*} = \eta$. Then $y_0, y_1 \in \partial B^{t^*} \cap \partial B^1$. As $\eta \subset cl(B^0)$ it follows that $t^* \in \{0, 1\}$. Since one of the points y_0, y_1 is in B^0, then $t^* = 1$. Contradiction. □

THEOREM 6.6 *Let $A \subset cl(B_R)$ be a regular plane compact set such that ∂A is connected and the counterclockwise oriented curve η with support ∂A has curvature greater or equal than $-1/R$. Then A has the property of the R-exterior ball.*

Proof. Let $z_0 \in \partial A$ and $n_A(z_0)$ be the exterior normal to A at z_0. Let

$$B^1 = B(z_0 + Rn_A(z_0)), \quad B^0 = B_R.$$

In what follows ∂B^1 will be clockwise oriented and $arc(a, b)$ will be the shorter arc on ∂B^1 from a to b, a, b are points on ∂B^1.

If $z_0 \in \partial B^0$, then $\eta \subset cl(B^0)$ is tangent to ∂B^0; as A and $cl(B^0)$ have the same outer normal vector, then $B^1 \cap A = \emptyset$ and B^1 R-supports A at z_0.

Let $z_0 \in B^0$. Let us notice that if $z \in cl(B^0) \cap cl(B^1)$, then $|z - z_0| < 2R$. If $A \subset (B^1)^c$, then A has the property of the exterior ball at z_0. To prove this fact it will be shown that there exist two points $z_0^+, z_0^- \in \eta \cap \partial B^1$ (z_0^+, z_0^- possibly coinciding with z_0), such that η is the union of $arc(z_0^-, z_0^+)$ and a regular curve $\tilde{\eta}$, with end points z_0^+, z_0^-, where

$$\tilde{\eta} \setminus \{z_0^-, z_0^+\} \subset (cl(B^1))^c.$$

Let (x, y) be a Cartesian coordinate system centered at z_0, with y-axis in the opposite direction of $n_A(z_0)$ and the x axis in the direction of the tangent vector to η at z_0. Let U be a neighborhood of z_0 such that $\eta \cap U$ is the graph of a function $y = f(x)$ and $\partial B^1 \cap U$ is the graph of $y = g(x)$.

If $\eta \cap U$ contains a point $z \in cl(B^1) \cap cl(B^0) \setminus \{z_0\}$ then, for a suitable $0 < l < R$, either $z = (l, f(l))$ or $z = (-l, f(-l))$. Then f, g satisfy the assumptions of Lemma 6.3. Therefore $\eta_{z,z_0} \cap U \subset \partial B^1$ ($\eta_{z_0,z} \cap U \subset \partial B^1$).

Let ξ be the maximal closed connected component of $\eta \cap cl(B^1)$ containing z_0. As $\xi \subset cl(B^1) \cap cl(B^0)$ then diam $(\xi) < 2R$; then ξ is an arc on ∂B^1 shorter than πR, with end points z_0^-, z_0^+, where $z_0^- \prec z_0 \prec z_0^+$ on $\partial B^1 \cap cl(B^0)$; it can be $z_0 = z_0^-$, $z_0 = z_0^+$, moreover by the regularity of η both $z_0^+, z_0^- \notin \partial B^0$.

Let $\tilde{\eta} = \eta \setminus \xi$ oriented accordingly to the counterclockwise orientation of ∂A. Let

$$W^+ = \{w \in \tilde{\eta} : z_0^+ \prec w, \tilde{\eta}_{z_0^+,w} \setminus \{z_0^+, w\} \subset (cl(B^1))^c\}; \tag{39}$$

$$W^- = \{w \in \tilde{\eta} : w \prec z_0^-, \tilde{\eta}_{w,z_0^-} \setminus \{z_0^-, w\} \subset (cl(B^1))^c\}. \tag{40}$$

The above argument shows that W^+, W^- are non empty sets. Let w^+ the supremum of W^+ (w^- the infimum of W^-) accordingly to the ordering of $\tilde{\eta}$. Then $\{w^+, w^-\} \subset \partial B^1$ and $\tilde{\eta}_{z_0^+,w^+}, \tilde{\eta}_{w^-,z_0^-}$ are subsets of $\tilde{\eta}$. If $w^+ = z_0^-$ then also $w^- = z_0^+$ and vice versa, moreover in this case

$$\tilde{\eta} \setminus \{z_0^-, z_0^+\} \subset (cl(B^1))^c$$

and the thesis holds.

Let us show first that $w^+ \neq z_0^-$, $w^- \neq z_0^+$ cannot hold. Let $\partial B^1 \cap \partial B^0 = \{u^-, u^+\}$, with $u^- \prec z_0^- \prec z_0^+ \prec u^+$ on $\partial B^1 \cap cl(B^0)$.

Let us show that $w^+ \in arc(z_0^+, u^+)$ ($w^- \in arc(u^-, z_0^-)$) cannot hold. As B^0, B^1 and $\eta_{z_0^+,w^+}$ satisfy the hypothesis of Lemma 6.5 with $y_0 = z_0^+, y_1 = w^+$, that would imply $\eta_{z_0^+,w^+} \subset \partial B^1$. This fact would contradict the maximality property of z_0^+ (similar procedure for w^-). The remaining case would be

$w^+ \in arc(u^-, z_0^-)$ and $w^- \in arc(z_0^+, u^+)$. This would imply that $\tilde{\eta}_{z_0^+, w^+}$ and $\tilde{\eta}_{w^-, z_0^-}$ should cross and that η is not a simple curve. This is impossible. □

The R-exterior ball property cannot hold for a set A without suitable topological assumptions. As example let us consider two concentric disks $D(O,R), D(O,2R)$ centered at the origin O. Let V a convex angle of vertex O with amplitude $\varepsilon > 0$. Let $U = D(O,2R) \setminus (D(O,R) \cup V)$. U is a regular domain (excepted four points) which can be modified in a neighborhood of this four points into a smooth domain A such that ∂A has the curvature greater or equal than $-1/R$ at each point. It easy to see that each B_R ball with boundary through a point on $\partial A \cap \partial V$ meets the interior of A for ε small enough.

The assumption that ∂A is connected is necessary too. Let us consider as A the union of two disjoint small circles contained in B_R. The counterclockwise oriented boundary of the circles have positive curvature but the R-exterior ball property does not hold.

In a forthcoming work [8] it will be shown that Theorem 6.6 is sharp. If $A \subset cl(B_{R+\varepsilon})$, with $\varepsilon > 0$ the result may not hold.

Theorem 6.7 *Let $\Omega \subset D_R$. Let the curvature of the level lines $\{x \in \Omega : u(x) = l\}$ (counterclockwise oriented), with $l > \min_\Omega u$, greater or equal than $-\frac{1}{R}$. Then the level sets of u have the property of the R-exterior ball and its steepest descent curves are R-curves.*

Proof. From the previous theorem, applied to each set $A = \Omega_l$, it follows that the level sets of u have the property of the R-exterior ball; then, by Theorem 6.1 the steepest descent curves of u are R-curves. □

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